Exercise 1: The strain energy density is given by,

$$W(\varepsilon_{ij},\sigma_{ij}) = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \qquad \text{(a)}$$

Use the Hook's law to express the energy density

(1) in terms of stresses,

(2) in terms of strains.

(3) take the derivative of the energy with respect to strain, respectively, stress to obtain the stresses and strains.

Exercise 2: Consider the Hook's law in index form (numbers refer to the book by Botsis & Deville and are used here for convenience),

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \qquad (6.174)$$

and the following definitions for the deviatoric stresses and strains:

$$\sigma_{ij} = \sigma_{ij}^d + \sigma_0\delta_{ij}, \quad \sigma_0 = \frac{1}{3}\sigma_{kk}, \quad \sigma_{ij}^d = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} \qquad (6.175)$$

$$\varepsilon_{ij} = \varepsilon_{ij}^d + \varepsilon_0\delta_{ij}, \quad \varepsilon_0 = \frac{1}{3}\varepsilon_{kk}, \quad \varepsilon_{ij}^d = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij} \qquad (6.176)$$

1. Show that (a) is equivalent to,

   a. $\sigma_{ij}^d = 2\mu\varepsilon_{ij}^d$ and $\sigma_0 = 3K\varepsilon_0$ (6.177)

2. Show that the principal axes of the stress and strain tensors coincide.

3. Show that the strain energy density can be expressed as the sum deviatoric and volumetric components,

   $$W(\varepsilon_{ij}) = \frac{\lambda}{2}\varepsilon_{ii}\varepsilon_{kk} + \mu\varepsilon_{ij}\varepsilon_{ij} = \frac{9}{2}K\left(\varepsilon_0\right)^2 + \mu\varepsilon_{ij}^d\varepsilon_{ij}^d = W_p(\varepsilon_{ij}) + W_d(\varepsilon_{ij}). \qquad (6.178)$$

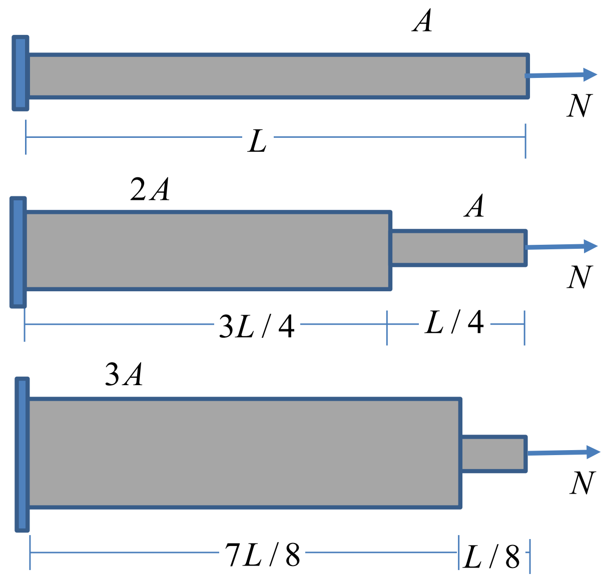
   where $K = (3\lambda + 2\mu)/3$ is the bulk modulus.

4. Show that the stability condition $W(\varepsilon_{ij}) > 0 \;\; \forall\varepsilon_{ij} \neq 0$ amounts to $K > 0, \mu > 0$.

Exercise 3: Three bars of different geometries as shown in the figure below are subjected to the same force *N*.

1. Compare the energies stored in each bar (*A* indicates area and the elastic modulus is known).

2. Calculate the maximum stress in each bar.



Exercise 4: A rectangular parallelepiped metallic bloc with dimensions $L_1 = 250$ mm, $L_2 = 200$ mm, $L_3 = 150$ mm is subjected to stresses $\sigma_1 =$ - 60 MPa, $\sigma_2 =$ - 50 MPa, $\sigma_3 =$ - 40 MPa. The mechanical properties are $E = 250$ GPa, $\nu = 0.3$.

Calculate:

(1) the changes in length L_1, L_2, L_3,

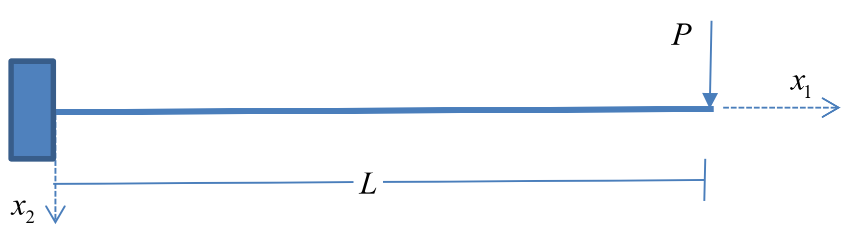
(2) the changes in its volume and

(3) the strain energy density.

Exercise 5: Use the principle of virtual work to determine the deflection at the free end of the cantilever beam shown in the Figure below. Use as deflection shape the function and consider only the energy due to bending. The bending stiffness is known *EI*.

$$u_2(x_1) = \frac{a x_1^2}{2L^3}(3L - x_1) \qquad \text{(a)}$$

where *a* is a constant.



Exercise 6: A simply supported beam is loaded as shown in the Figure. Determine the deflection at the load application point in the direction of the load. Use the Rayleigh-Ritz method and assume

$$u_2(x_1) = a x_1 (L - x_1) \qquad \text{(a)}$$

where a is to be determined. The bending stiffness EI is known. Consider only the energy due to bending.